Simple measure for complexity

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(Received 29 July 1998)

A measure of ''complexity'' is proposed, based on appropriately defined notions of ''order'' and ''disorder,'' which has a considerable degree of flexibility in its dependence on these concepts. The possible functional dependencies which result encompass those of many earlier definitions of complexity. The proposed measure is in principle easy to calculate and has the property of an intensive thermodynamic quantity. With appropriate choices of parameters it behaves similarly to ''effective measure complexity'' for the logistic map. It is also a generalization of the ''normalized complexity'' of López-Ruiz *et al.*, but does not suffer from ''over-universality.'' [S1063-651X(99)03702-2]

PACS number(s): 05.20.−y, 05.90.+m

I. INTRODUCTION

Many authors have commented on and discussed the multiplicity and variety of definitions of complexity in the literature ([1–6], and references therein). Current definitions may be divided into three broad categories. First are those definitions which take complexity to be a monotonically increasing function of disorder; examples are algorithmic complexity [7,8] and the various entropies [9–11]. As numerous as definitions of the first category are those in which complexity is a convex function of disorder; i.e., complexity is a minimum for both completely ordered and completely disordered systems, and a maximum at some intermediate level of disorder or order. To this category belong logical depth [12] and thermodynamic depth [1]. Finally, there are some definitions which take complexity to be loosely the same as order; these identify complexity broadly with the level of self-organization and self-organization with order (see [3]). The three categories of complexity are summarized schematically in Fig. 1.

In recent studies [13–15] we have introduced another measure of complexity which, depending on the choice of parameters, may display the behavior of any of the three categories of Fig. 1. The impetus for developing this measure was provided by the question of whether biological and other complex systems evolve so as to optimize complexity, or putting it the other way around, how such systems would evolve if complexity were to be maximized. An approach to this problem can be based on optimization theory [16], but to make this approach tractable one needs an easily evaluated measure for complexity. While many of the previously proposed definitions are intuitively appealing, they often suffer from the disadvantage of being difficult to compute. An example is algorithmic complexity [7,8], the length of the shortest possible program necessary to reproduce a given object. The difficulty arises in proving that a given program is indeed the shortest. In contrast, our measure for complexity can be calculated easily. Notwithstanding the ease of computation, our measure behaves similarly to the effective measure complexity [17] for the logistic map.

An additional advantage of our complexity measure is that it is independent of size effects in a manner similar to intensive thermodynamic quantities such as temperature or pressure. While it is possible to argue that larger systems are necessarily more complex simply by virtue of their greater size, we seek a complexity measure which does not increase simply because a system becomes larger. This advantage is not unique to our proposal; algorithmic complexity [8] as well as some other earlier proposals for complexity share it [18,19].

Feldman and Crutchfield [20] have independently arrived at our complexity measure for one special set of parameters by a procedure they refer to as ''repairing nonextensivity'' of still another measure, ''normalized complexity,'' put forward by López-Ruiz *et al.* [21]. Indeed, for this parameter set the measure of López-Ruiz *et al.* will be shown to be an approximation to our measure. Feldman and Crutchfield criticize the repaired measure as being ''over-universal''; i.e., it has the same dependence on disorder under all conditions. We will show that this is not the case.

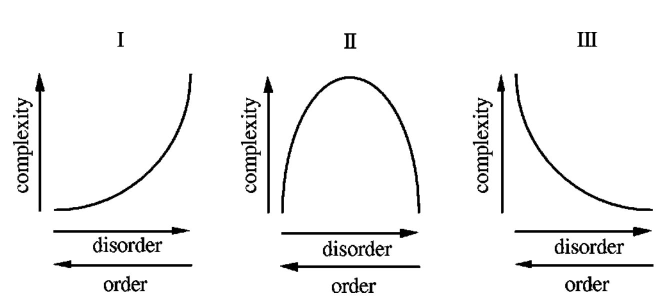

FIG. 1. The three categories of complexity as a function of disorder.

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II. MEASURES OF DISORDER, ORDER, AND COMPLEXITY

A. Disorder and order

Since we will express our complexity measure in terms of disorder and order, we first need to present our measures for the latter. As pointed out earlier by one of us [22–24], definitions of order and disorder can suffer from the same problem regarding system size as those of complexity. Often entropy is taken to be an appropriate measure of disorder. However, this tacitly assumes that the size of the system, as measured by the number of states available to it, does not change. In fact, if the number of states of the system increases then the entropy and therefore the disorder of the system will also increase for no other reason than the increase in the number of states. To circumvent this problem, it has been proposed [22–24] that ''disorder'' be defined as

$$\Delta \equiv S/S_{\max}, \tag{1}$$

where S is the Boltzmann-Gibbs-Shannon (BGS) entropy [9,10]

$$S \equiv -k \sum_{i=1}^{N} p_i \ln p_i . \tag{2}$$

p_i is the probability of state i of the N states available to the system, and k is the Boltzmann constant, appropriate to a physical system. It can be replaced by any other appropriate constant for other types of systems (e.g., ln2 for information systems) or omitted for a dimensionless entropy. (When referring to specific measures of disorder, order, or complexity, we enclose these words in quotation marks to distinguish the measures from the general concepts.)

In the simplest case, that of an isolated system, the entropy maximum occurs at the equiprobable distribution, $p_i = 1/N$, $1 \leqslant i \leqslant N$, yielding

$$S_{\max} = k \ln N \tag{3}$$

as the maximum possible entropy. However, in many cases this maximum is not attainable, for example in grand canonical equilibrium, when the mean energy and mean number of particles are fixed. Consider a system of particles with a given total energy and assume that it is not at equilibrium. We now isolate the system and let it relax to equilibrium. Since the system is now isolated neither the total number of particles nor the total energy can change. The equilibrium state is characterized as that state with the maximum entropy *subject to the constraints on particle number and total energy*. This maximum entropy will be less than that corresponding to the equiprobable distribution, and is the appropriate maximum entropy to be used in the definition of disorder for this system. This was done in our study of a nonequilibrium ideal gas [13]. When such constraints are absent, as in the microcanonical ensemble, and the entropy is maximized, one naturally finds the larger entropy of Eq. (3).

''Order'' is defined as

$$\Omega \equiv 1 - \Delta. \tag{4}$$

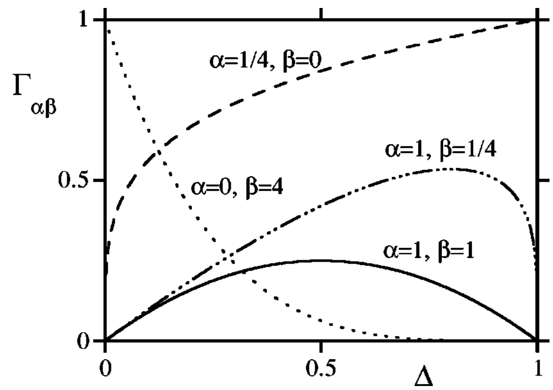

FIG. 2. ''Complexity'' $\Gamma_{\alpha\beta}$ as a function of ''disorder'' Δ for α, $\beta \geqslant 0$. Shown are one case with $\Gamma_{\alpha\beta}$ increasing monotonically with Δ ($\alpha = 1/4$, $\beta = 0$), one case with $\Gamma_{\alpha\beta}$ decreasing monotonically with Δ ($\alpha = 0$, $\beta = 4$), and two cases where $\Gamma_{\alpha\beta}$ shows a convex dependence on Δ ($\alpha = 1$, $\beta = 1/4$; $\alpha = \beta = 1$).

Both ''order'' and ''disorder'' lie between 0 and 1 and are freed from the size effect. It is now possible that entropy and ''order'' increase together as the size of the system increases. This has been shown to occur for certain simple, well-defined examples, as well as for certain aspects of the universe, including evolution and phylogeny [13–15,22–25].

B. Complexity

For complexity measures of category I (Fig. 1) complexity is a monotonically increasing function of disorder. Since the definition [Eq. (1)] of disorder has advantages over entropy as a measure of disorder, any monotonically increasing function of disorder as defined here would also be an appropriate measure of complexity for this category. For the third category of complexity measures, where complexity is taken to increase with order, any increasing function of order [Eq. (4)] would similarly be an appropriate measure. For category II complexity measures, where complexity is a convex function of disorder, one of the simplest possible functional forms for complexity is the product of ''order'' and ''disorder,'' $\Delta\Omega = \Delta(1-\Delta) = \Omega(1-\Omega)$. All three categories of complexity measures can thus be subsumed by a measure of the form

$$\Gamma_{\alpha\beta} \equiv \Delta^{\alpha}\Omega^{\beta} = \Delta^{\alpha}(1-\Delta)^{\beta} = \Omega^{\beta}(1-\Omega)^{\alpha}, \tag{5}$$

which we call the ''simple complexity of disorder strength α and order strength β.'' When β vanishes and $\alpha > 0$, ''complexity'' is an increasing function of ''disorder,'' and we have a measure of category I. When α vanishes and $\beta > 0$, ''complexity'' is an increasing function of ''order,'' and we have a definition of category III. When both α and β are nonvanishing and positive, ''complexity'' vanishes at zero ''disorder'' and zero ''order,'' and has a maximum of

$$(\Gamma_{\alpha\beta})_{\max} = \alpha^{\alpha}\beta^{\beta}/(\alpha+\beta)^{(\alpha+\beta)} \tag{6}$$

at

$$\Delta = \alpha/(\alpha+\beta), \quad \Omega = \beta/(\alpha+\beta).$$

Several cases for both α and β non-negative are shown in Fig. 2. The qualitative behavior if either α or β (or both) is negative, as well as possible normalizations and transformations to extensive quantities, are treated briefly in the Appendix.

When α or β are both positive $\Gamma_{\alpha\beta}$ can be interpreted in terms of the popular notion that complex systems are often nonequilibrium systems. As can be inferred from the discussion of the maximum entropy in the preceding section, $\Omega = 1 - S/S_{\max} = 1 - S/S_{eq} = (S_{eq} - S)/S_{eq}$ is a measure of the distance from equilibrium; in fact, Ebeling and Klimontovich [26] introduced $S_{eq} - S$ as a correct measure of distance from equilibrium where the equilibrium state is achieved by the isolation procedure described above. Thus, for nonequilibrium systems, our simple measure of complexity [Eq. (5)] is a function of both the ''disorder'' of the system and its distance from equilibrium. ''Complexity'' vanishes either if the system is at equilibrium, implying maximum ''disorder,'' or if it is completely ordered, implying maximal distance from equilibrium. Only if the system has some less than maximal ''order'' and is not at equilibrium, does it possess a nonvanishing level of ''complexity.''

III. THE LOGISTIC MAP—AN EXAMPLE

To illustrate the properties of our proposals for measures of complexity we choose the logistic map as a concrete example:

$$x_{n+1} = r x_n (1 - x_n). \tag{7}$$

The logistic map is a well-studied simple system which displays a rich variety of behaviors. Depending on the value of the parameter r the map may show a stable point, oscillations, period doubling, or chaos. In our calculations for fixed r we discarded the first 10 000 points and calculated an additional 100 000. The values of x were then assigned to 1024 bins of equal size between 0 and 1. The probabilities that the points are in the various 1024 bins were then used to calculate the entropies and other values. r was varied between 3.5 and 4.0 in steps of 0.001.

''Disorder'' is proportional to the entropy here, since the number of possible states of the system, just the number of bins, does not change from one value of r to another. The refinement of the calculations (1024 bins) is enough that Δ behaves identically to the Rényi dimension $D^{(1)}$ ([5]—Fig. 6). Thus ''disorder'' itself does not lead to any new results, since entropy has often been proposed as a measure for complexity.

The behavior of the ''complexity'' Γ_{11}, shown in Fig. 3, is more interesting. It behaves similarly to the ''effective measure complexity'' (EMC) of Grassberger [17], as calculated by Wackerbauer *et al.* ([5]—Fig. 10). It has the same general form; major maxima as well as less major ones occur at the same values of r, as do the plateaus. Different are the relative values of the peaks. Why Γ_{11} behaves similarly to EMC is not readily apparent; nor is the breadth of the class of systems for which this is the case. The two quantities are, after all, calculated in very different ways. Although both would be classified by Wackerbauer *et al.* as structural measures of complexity, EMC is defined in terms of the local slopes of the information entropy in an attempt to obtain a dynamic measure. Furthermore, EMC uses these slopes for

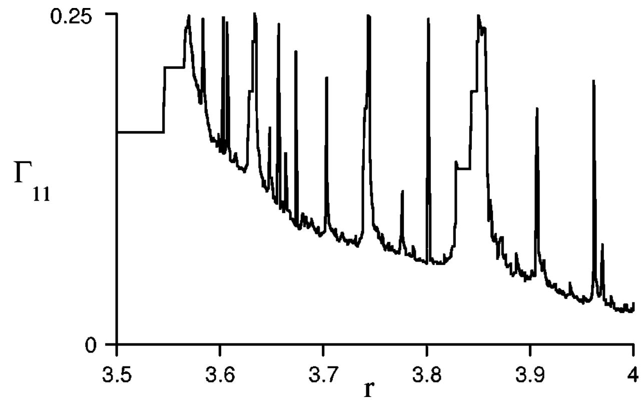

FIG. 3. ''Complexity'' Γ_{11} of the logistic map.

symbol sequences of length n and takes the limit as $n \to \infty$. (In symbolic dynamics symbols are assigned to bins, and sequences are obtained as the strings of symbols in consecutive steps.) Finally, in calculating EMC Wackerbauer *et al.* used a generating partition, i.e., one based on the dynamics of the logistic map, to obtain the binning. On the other hand, in calculating Γ_{11} we used a homogeneous partition (equal size bins), used effectively symbol sequences of length 1 to calculate the entropy itself, not the local slope, and did not need to take a limit.

Additonal calculations based on the Rényi entropy [11] of order 2, not the BGS entropy, were done. They showed qualitatively the same behavior as those based on the BGS entropy.

IV. RELATION TO SOME OTHER COMPLEXITY MEASURES

López-Ruiz *et al.* [21] have proposed a complexity measure which we now show is an approximation to our Γ_{11}. Their ''normalized complexity'' $\bar{C}$ is defined by

$$\bar{C} \equiv \Delta D \qquad \text{(López-Ruiz } et\ al.\text{)}, \tag{8}$$

where

$$D \equiv \sum_{i=1}^{N} \left(p_i - \frac{1}{N} \right)^2 \tag{9}$$

expresses the notion of ''disequilibrium'' of a system of N accessible states and measures the ''distance'' of a system state, given by the probabilities $p_1, p_2, \ldots, p_N$, from the system state of equiprobability, defined by $p_1 = p_2 = \cdots = p_N = 1/N$. It vanishes only in microcanonical equilibrium.

We develop a Taylor expansion of Ω around the equiprobable distribution, denoted by $(\ )_0$. Noting from Eqs. (1)–(4) that

$$\Omega = 1 + \frac{\sum_{i=1}^{N} p_i \ln p_i}{\ln N}, \quad \Omega_0 = 0 \tag{10}$$

the required expansion is

$$\Omega=\Omega_0+\sum_{i=1}^{N}\left(\frac{\partial\Omega}{\partial p_i}\right)_0\left(p_i-\frac{1}{N}\right)$$
$$+\frac{1}{2}\sum_{i=1}^{N}\left(\frac{\partial^2\Omega}{\partial p_i^2}\right)_0\left(p_i-\frac{1}{N}\right)^2+\cdots$$
$$=0+\left(\frac{1}{\ln N}-1\right)\sum_{i=1}^{N}\left(p_i-\frac{1}{N}\right)$$
$$+\frac{1}{2}\frac{N}{\ln N}\sum_{i=1}^{N}\left(p_i-\frac{1}{N}\right)^2+\cdots$$
$$=\frac{1}{2}\frac{N}{\ln N}D+\cdots. \tag{11}$$

As the second term of the expansion vanishes, we see that ''disequilibrium'' D is, to within a factor dependent on N, just a second-order expansion of Ω around the equiprobable distribution. Thus the ''normalized complexity'' $\overline{C}$ satisifies

$$\overline{C}=\Delta D\cong\Delta\frac{2\ln N}{N}\Omega=\frac{2\ln N}{N}\Gamma_{11}. \tag{12}$$

Let us also note that the criticism of a somewhat unconvincing maximum complexity $\overline{C}$, noted recently [27], does not apply to our Γ_{11}. Anteneodo and Plastino [27] found that $\overline{C}$ is maximum only for distributions $\{p_i\}$ where one $p_i=2/3$ and all other p_i are equal. On the other hand, our maximum complexity occurs at

$$\Delta=\frac{-\sum_{i=1}^{N}p_i\ln p_i}{\ln N}=\frac{1}{2} \tag{13}$$

and there are many distributions $\{p_i\}$ which can satisfy *this* constraint. We give just two of the simpler examples which yield maximum complexity, i.e., $\Delta=1/2$. In the first example, n of the states i have probability p^*, and the other $N-n$ states have probability $(1-np^*)/(N-n)$. For $N=100$ and $n=9$ we find $p^*\approx 0.109\,57$; this distribution can be realized in more than 10^{12} ways. The second example distribution has n_1 states of probability p^* and n_2 states of probability p^{**}; the other $N-n_1-n_2$ states have probability $(1-n_1p^*-n_2p^{**})/(N-n_1-n_2)$. Taking $N=100$ again and $n_1=n_2=5$, we find that $p^*=0.002$ and $p^{**}\approx 0.177\,05$ yield another set of distributions giving $\Delta=1/2$. These distributions can be realized in more than 10^{15} ways. On the other hand, low complexity situations lead to a comparatively small number of sets $\{p_i\}$. Thus $\Gamma_{11}=0$ implies one of only two types of distributions $\{p_i\}=\{1,0,\ldots,0\}$ or $\{p_i\}=\{1/N,1/N,\ldots,1/N\}$. The ordering of the states has been arranged here so as to assign the label $i=1$ to the state which has probability 1. For $N=100$ there are only a total of 101 distributions which lead to $\Gamma_{11}=0$.

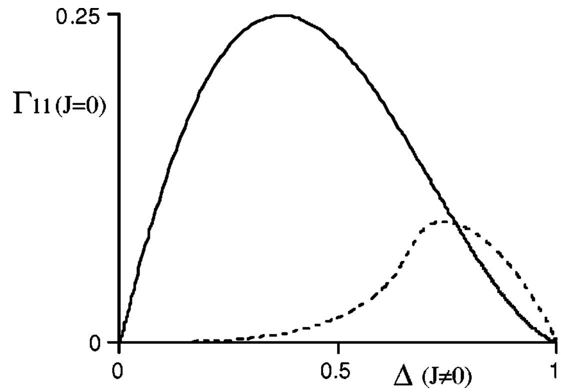

FIG. 4. The ''complexity''-''disorder'' diagram for simple spin systems. ''Complexity'' Γ_{11} is calculated in the absence of interactions and ''disorder'' in the presence of interactions. Solid line: ferromagnet (interaction parameter $J=1$, external field=0.3); dashed line: antiferromagnet (interaction parameter $J=-1$, external field=1.8).

Feldman and Crutchfield [20] have independently arrived at our Γ_{11} from $\overline{C}$ by a procedure they refer to as ''repairing nonextensivity.'' However, they criticize Γ_{11} as being ''over-universal''; i.e., it is uniquely determined by Δ. This criticism applies implicitly to our complexity measure for any values of α and β, and to any other measure of complexity which can be expressed solely in terms of disorder. Two complexity measures which Feldman and Crutchfield consider superior in that they are not ''over-universal'' are the excess entropy (effective measure complexity [17]) and the statistical complexity C_μ [28,29]. They studied these measures for simple one-dimensional spin systems with interaction parameter J and found that $C_\mu=\Delta_{J=0}$ and the effective measure complexity is given by $\Delta_{J=0}-\Delta_J$, where $\Delta_{J=0}$ is (in our nomenclature) the disorder of the system in the absence of interactions, and Δ_J the disorder in the presence of interactions [30]. They found that the dependence of both C_μ and EMC on Δ_J varies with the value of J. It is in this sense that C_μ and EMC are not ''over-universal.''

Let us do a similar calculation here. We calculate Γ_{11} for the spin system in the absence of interactions $(J=0)$ and consider its dependence on disorder in the presence of interactions $(J\neq 0)$. Sample results are shown in Fig. 4. We see that the two curves, one for a ferromagnet and the other for an antiferromagnet, are different. Therefore our complexity measure is not ''overuniversal'' in the same sense as C_μ and EMC are not: ''complexity'' is obtained from ''disorder'' calculated under one set of conditions, and its dependence on ''disorder'' calculated under different conditions is considered.

V. DISCUSSION

We have proposed a definition of complexity which encompasses the qualitative behavior of most previous definitions. Depending on the values of the disorder and order strengths, it may increase monotonically with ''disorder,'' increase monotonically with ''order,'' or reach an extremum value at intermediate values of ''order'' and ''disorder.''

Furthermore, our definition is easy to calculate in principle and provides measures of complexity which are independent of extensive size effects. Atmanspacher *et al.* [6] have argued that measures of complexity can be classified as deterministic or statistical, corresponding broadly to our category I (complexity increases monotonically with disorder) and category II (complexity shows ''a globally convex behavior as a function of randomness''), respectively. They argue that measures of category I are first-order statistical measures, whereas those of category II are second-order measures. It would seem that, at least in simple cases, our proposed definition fits into their arguments in that when complexity shows nonmonotonic behavior it is a function of two (related) statistical measures, ''disorder'' and ''order.''

It is important that the ''complexity'' Γ_{11} behaves as effective measure complexity [17], one of the more frequently used measures of complexity, for the logistic map. Why this is the case will require further investigation. However, it is a significant finding since the ''complexity'' Γ_{11} is perhaps easier to fathom than effective measure complexity and since Γ_{11} is simpler to calculate than EMC.

These results negate what would seem to be reasonable contentions that complexity cannot have a simple measure and that it cannot be expressed simply in terms of entropy or disorder. Indeed $\Gamma_{\alpha\beta}$ is a simple quantity and is defined solely in terms of the entropy and system size. Nonetheless, it behaves as one would expect for the various complexity categories and even approximates the behavior of EMC well for the logistic map. Our results therefore support Simon [31], who argued that the description of complex systems need not be complex; i.e., complexity is not a conserved quantity.

The very simplicity of our complexity measure adds to its usefulness. If we were to work only with simple model systems, we might be able to make use of more involved complexity measures, but for real systems these will rarely be available. Consider biological systems, which are among those for which concepts such as complexity are thought to be important. Many of these systems are poorly understood, and we have limited information about them. We might count ourselves lucky if we have enough information to be able to make an estimate of the number of states available to such systems and the relative frequencies of the states. This would just suffice to calculate ''disorder,'' most likely at some coarse grained level of description, and through ''disorder'' our simple measure of complexity. However, at the present state of our knowledge we are far from having enough information to calculate many of the other complexity measures, such as effective measure complexity [17] or statistical complexity [28,29]. Nonetheless, many important questions, e.g., the suggestion that neutral evolution can occur only under conditions of isocomplexity [18,19], require that we be able to evaluate some measure of complexity if we are to begin to attack them.

ACKNOWLEDGMENTS

This work was supported by Grant No. 31-42069.94 from the Swiss National Science Foundation and Grant No. 93.0106 from the Swiss Federal Office for Education and Science within the scope of EU Contract No. CHRX-CT92-0007 (J.S.S.) and EU Contracts No. ICOP-DISS-2168-96 and No. ERB CH RXCT 92 0007 (P.T.L.)

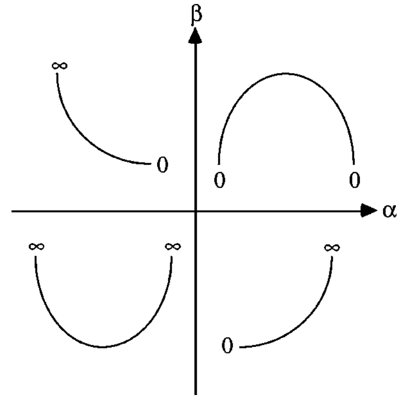

FIG. 5. The qualitative behavior of $\Gamma_{\alpha\beta}$ for the possible combinations of signs of α and β. All curves show ''complexity'' as it changes from zero ''disorder'' (on the left) to maximum ''disorder'' (on the right). It is assumed that both α and β are nonzero.

APPENDIX: POSSIBLE MODIFICATIONS OF OUR COMPLEXITY MEASURE

In actual cases one might expect α and β in Eq. (5) to be positive and nonzero, and this case can be shown to yield type II complexity (see Fig. 2). It is of course possible in principle that either α or β, or both, be negative. The qualitative behavior of $\Gamma_{\alpha\beta}$ for the four possible combinations of signs of α and β are shown in Figs. 5 and 6.

We also wish to hint at two possible further modifications of Eq. (5).

1. Normalization

For the cases where α and β are of the same sign and there is an extremum in the relation between ''complexity'' and ''disorder,'' it may be convenient to normalize ''complexity'' to its value at the extremum. For these cases we define the normalized ''complexity'' as

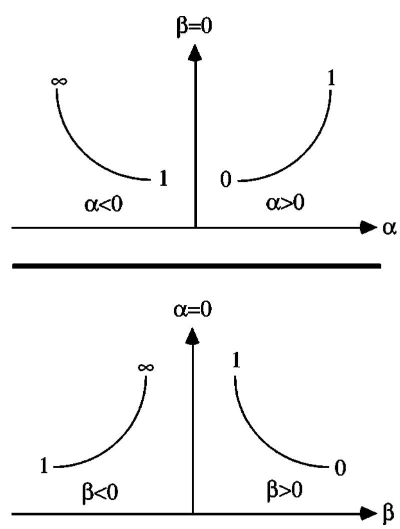

FIG. 6. The same as Fig. 5 but if either α or β is zero.

$$\hat{\Gamma}_{\alpha\beta} \equiv \frac{(\alpha+\beta)^{(\alpha+\beta)}}{\alpha^\alpha \beta^\beta} \Delta^\alpha \Omega^\beta = \frac{(\alpha+\beta)^{(\alpha+\beta)}}{\alpha^\alpha \beta^\beta} \Delta^\alpha (1-\Delta)^\beta = \frac{(\alpha+\beta)^{(\alpha+\beta)}}{\alpha^\alpha \beta^\beta} \Omega^\beta (1-\Omega)^\alpha. \tag{A1}$$

Then the limits on the normalized ''complexity'' are

$$0 < \hat{\Gamma}_{\alpha\beta} < 1 \, (\alpha, \beta > 0),$$
$$1 < \hat{\Gamma}_{\alpha\beta} < \infty \, (0 > \alpha, \beta). \tag{A2}$$

In the latter case, it might be more appropriate to take the reciprocal of complexity, which could be called ''simplicity,'' to arrive at

$$0 < \widehat{\sum}_{\alpha\beta} < 1 \quad \left(0 > \alpha, \beta; \widehat{\sum}_{\alpha\beta} \equiv 1/\hat{\Gamma}_{\alpha\beta} \right).$$

2. Related extensive complexities

Although we have maintained up to this point that complexity should not in general be an extensive quantity, occasionally it may be desirable to have an extensive complexity measure. If this is the case, one can simply remove the normalization factor $S_{\max}$ from the definition of ''order'' or ''disorder'' or both [Eqs. (1)–(4)] to obtain the following extensive ''complexities'':

$$G_{\alpha\beta} \equiv \begin{cases} S^\alpha \Omega^\beta = S^\alpha (1 - S/S_{\max})^\beta \\ \Delta^\alpha (S_{\max} - S)^\beta = (S/S_{\max})^\alpha (S_{\max} - S)^\beta \\ S^\alpha (S_{\max} - S)^\beta, \end{cases} \tag{A3}$$

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